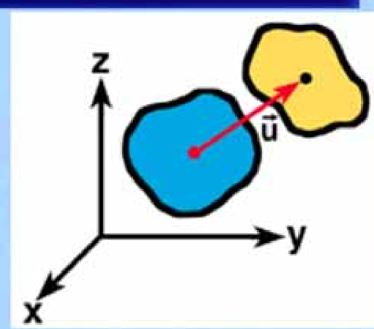
2.2 Kinematics of Deformation

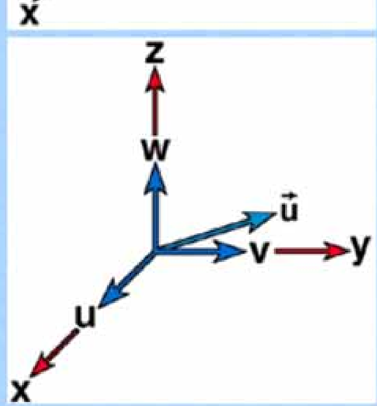
- 2.2.1 Displacement Vector at a Point
- 2.2.2 Deformation of a Deformable Body
- 2.2.3 Strain-Displacement Relationships
- 2.2.4 Analysis of Strain
  - 2.2.4.1 Transformation of Strain Components
  - 2.2.4.2 Principal Strains and Principal Directions
  - 2.2.4.3 Plane Strain
  - 2.2.4.4 Mohr's Circle Representation of Plane Strain
- 2.2.5 Strain Measurements
- 2.2.6 Strain Compatibility Relations

Kinematics of Deformation

- Kinematics is the branch of mechanics which deals with the motion without reference to force or mass
- Displacement is any change in the configuration of the body
- Displacement vector of a point

$$\vec{u} = \begin{bmatrix} u & v & w \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}$$

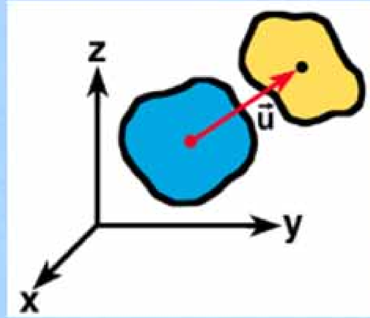


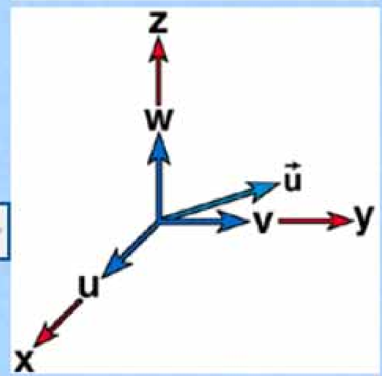


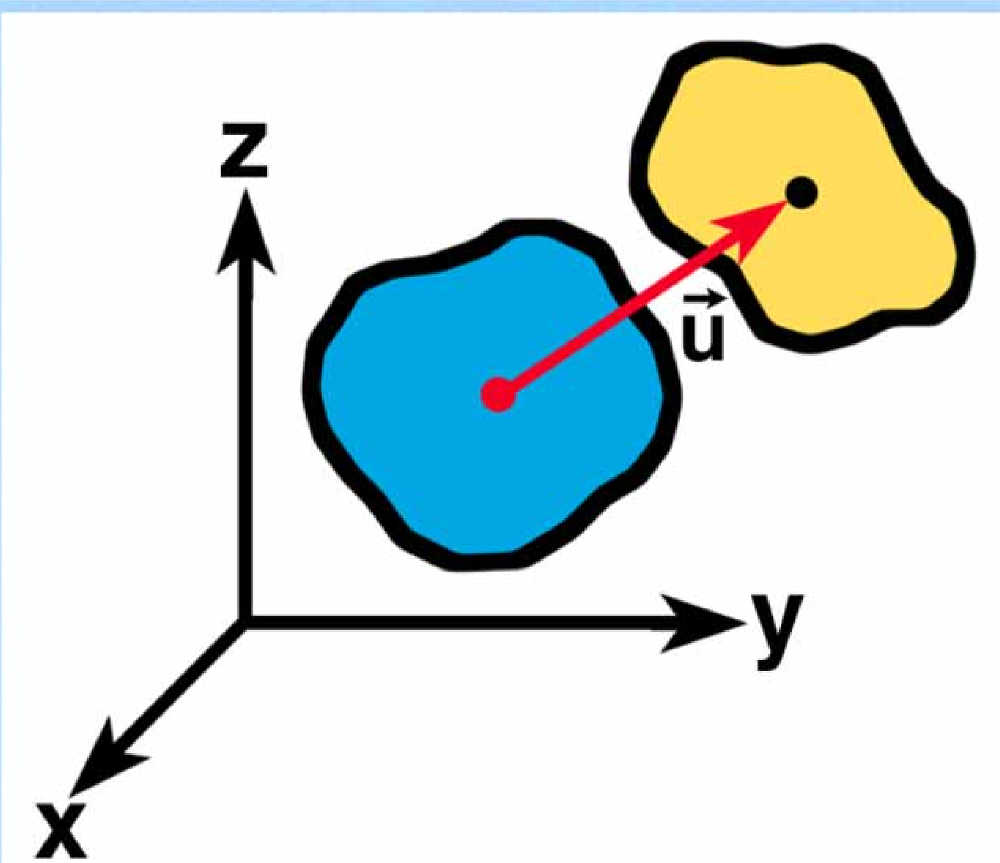
Kinematics of Deformation

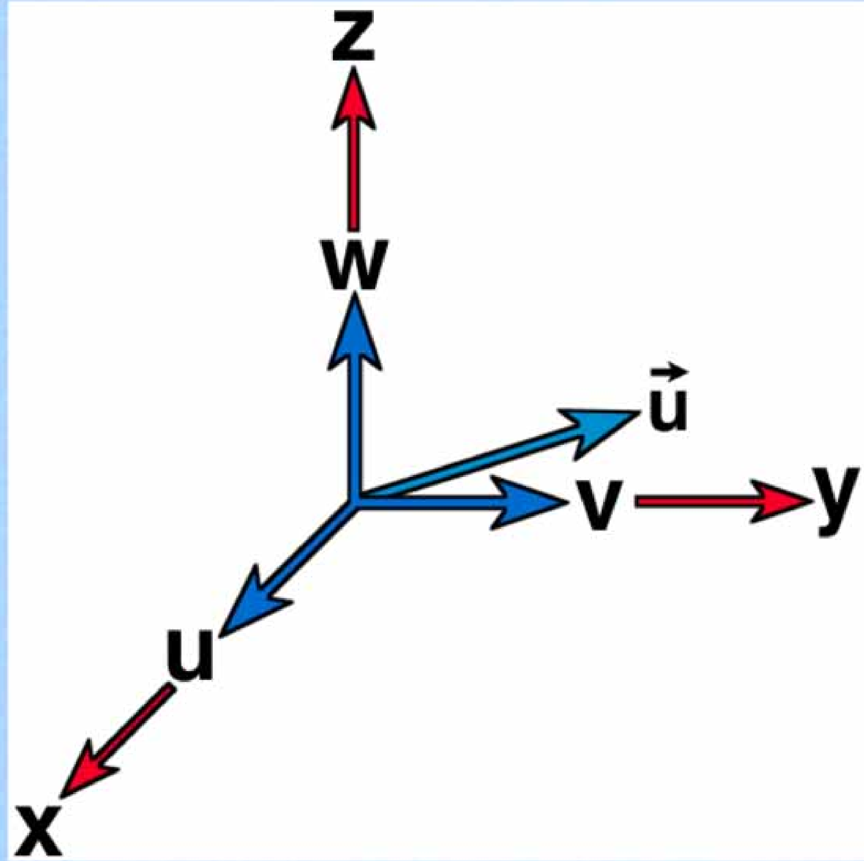
- Kinematics is the branch of mechanics which deals with the motion without reference to force or mass
- Displacement is any change in the configuration of the body
- Displacement vector of a point

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} u(x,y,z,t) \\ v(x,y,z,t) \\ w(x,y,z,t) \end{Bmatrix} \quad , \quad t = \text{time}$$





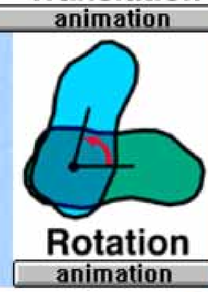


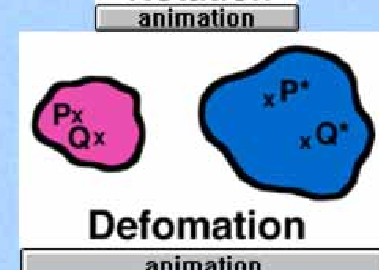


Displacement Vector at a Point

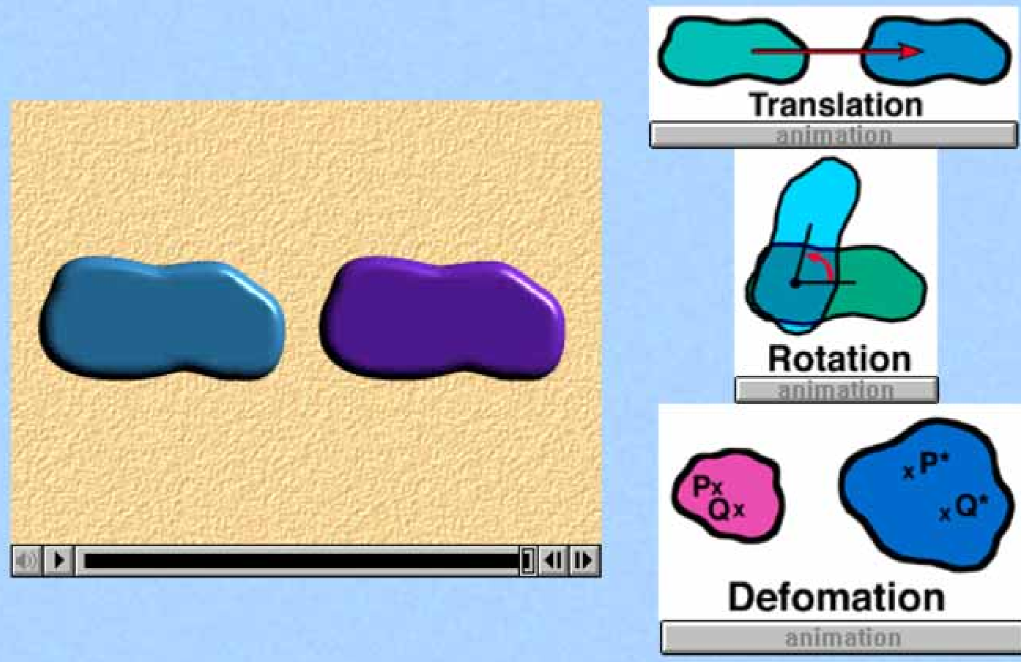
Displacement is associated with two phenomena:

- Rigid body motion
  - Translation
  - Rotation
- Deformation
- Change in the distance between material points and/or shape of body
- Measured by strain vector (or strain components) at a point

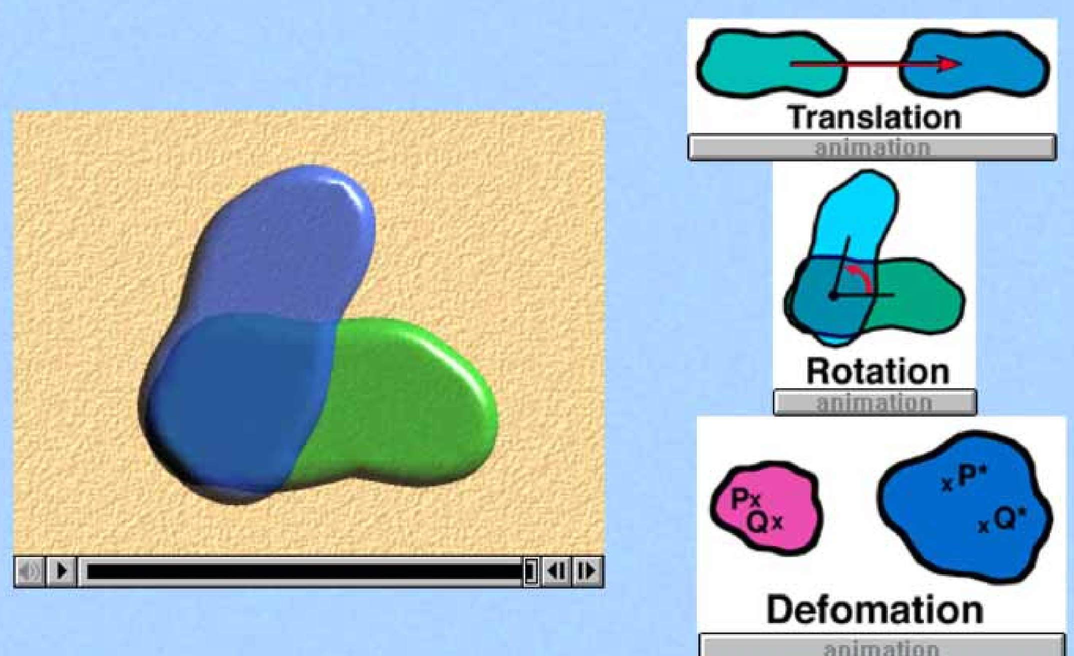




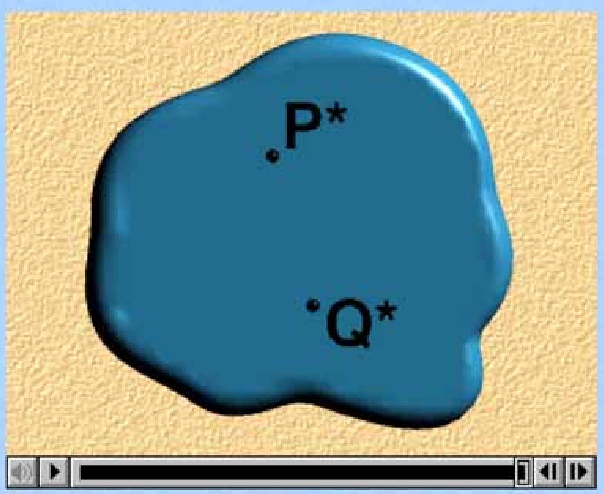
Displacement Vector at a Point

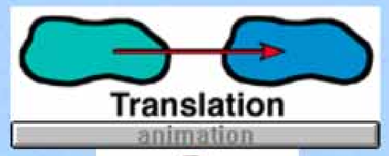


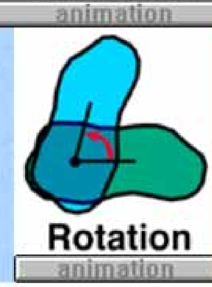
Displacement Vector at a Point

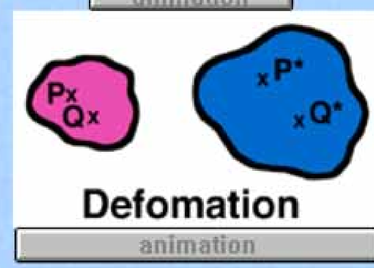


Displacement Vector at a Point







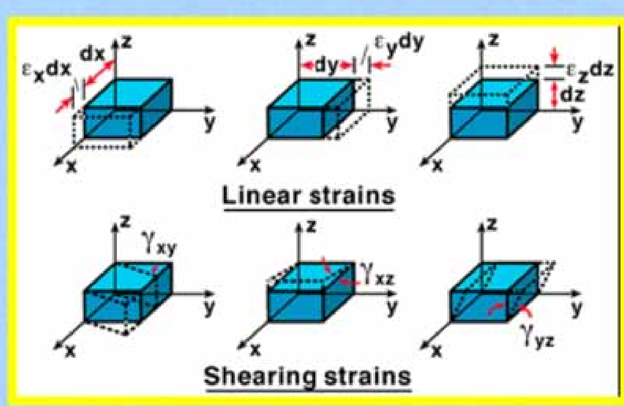


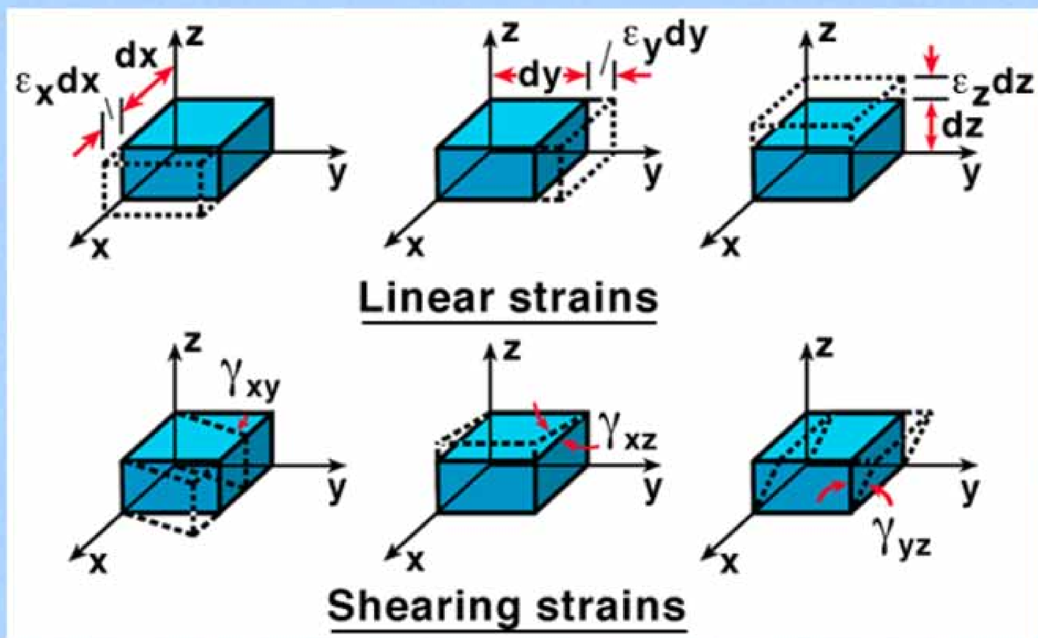
Deformation of a Deformable Body

Consider an elemental volume at a point, with extent dx, dy, dz in the x, y, z coordinate direction.

Deformation of the elemental volume consists of:

- Linear (or extensional) strains - measuring the change in the linear dimensions
- Shearing strains - measuring the change in the angles between the sides

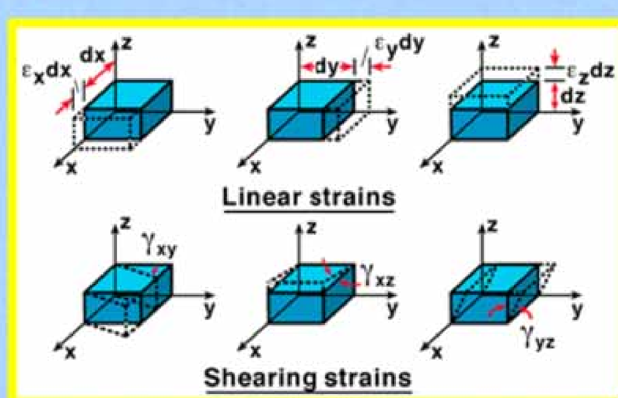




Deformation of a Deformable Body

Deformation of the elemental volume consists of:

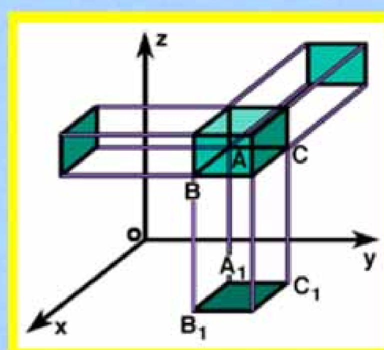
- Linear (or extensional) strains - measuring the change in the linear dimensions
- Shearing strains - measuring the change in the angles between the sides
- Curvature of sides - usually small and is neglected in a first approximation

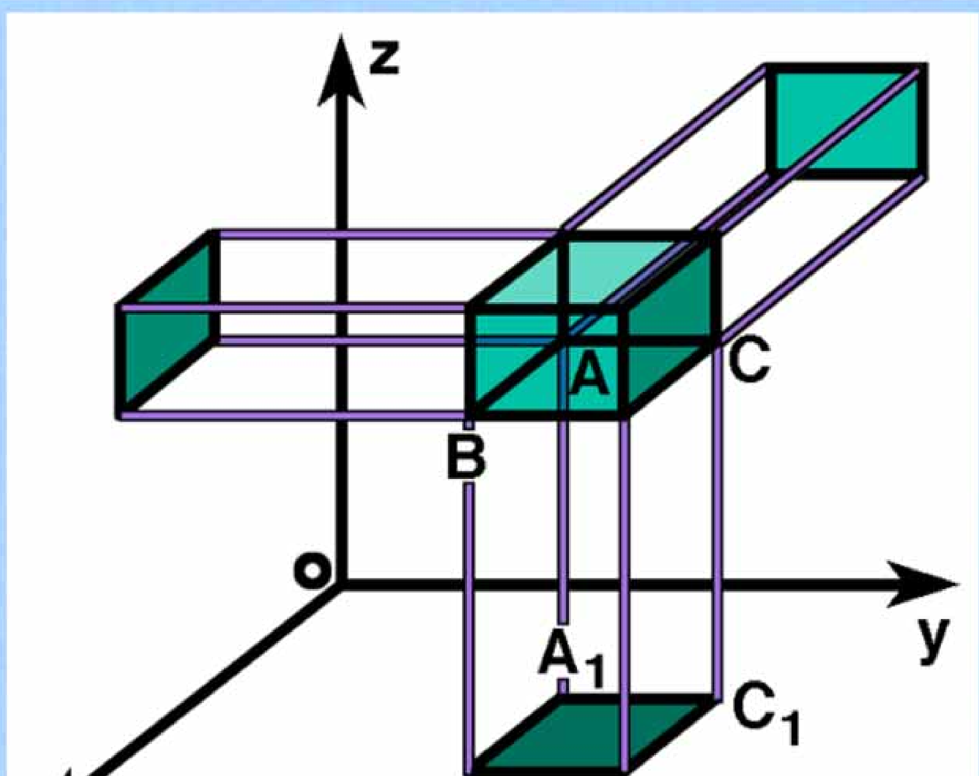


Strain-Displacement Relationships

- Consider the projections of the elemental volume (at a point) on the coordinate planes

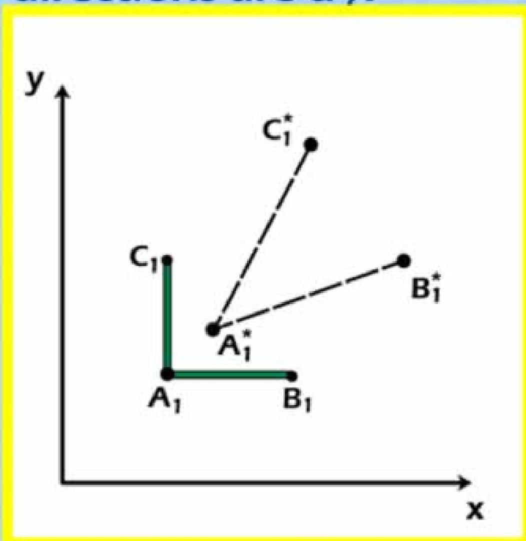
- The projections of the two lines AB, AC on the xy plane is A_1B_1, A_1C_1 where $A_1B_1 = dx$, $A_1C_1 = dy$

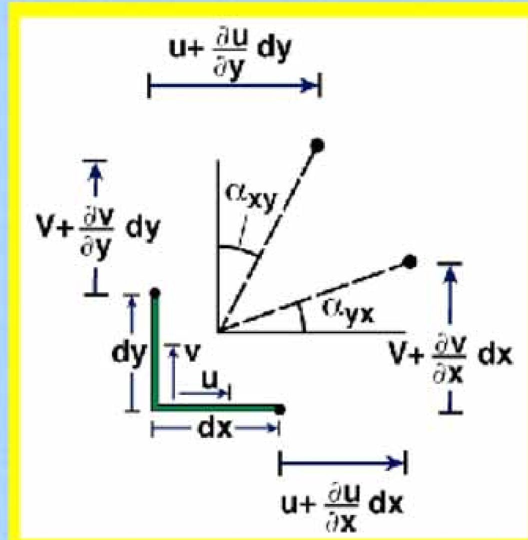


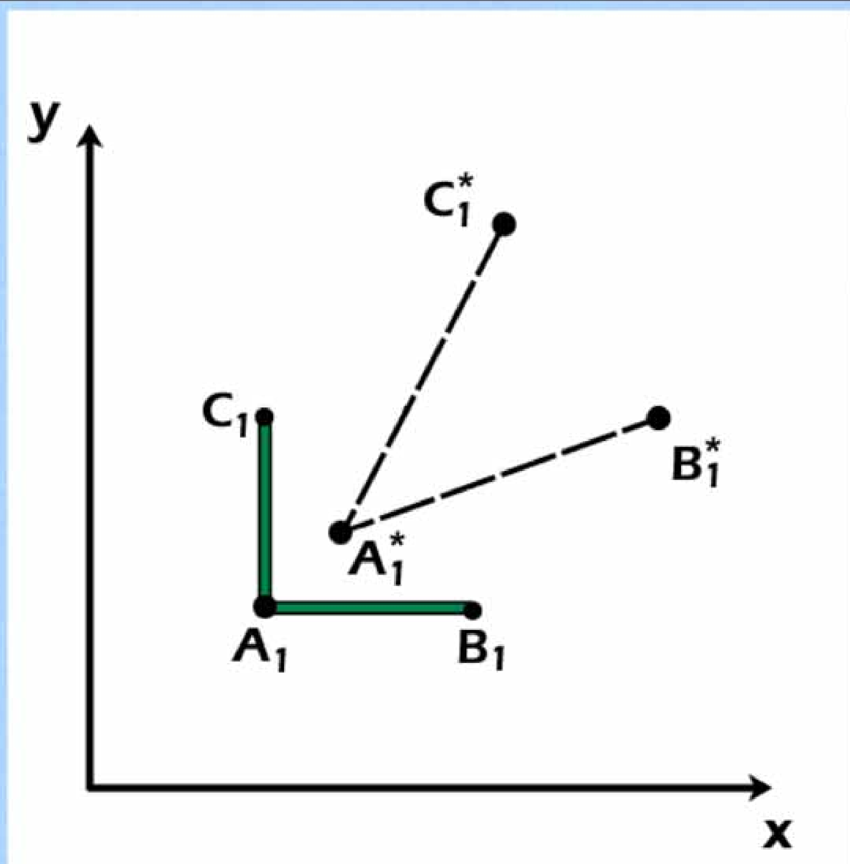


Strain-Displacement Relationships

If the displacements of point A_1 in the x ,y directions are u ,v







Strain-Displacement Relationships

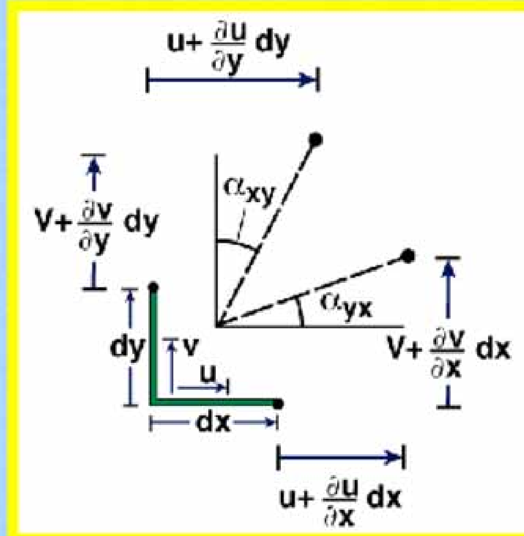
If the displacements of point A_1 in the x ,y directions are u ,v

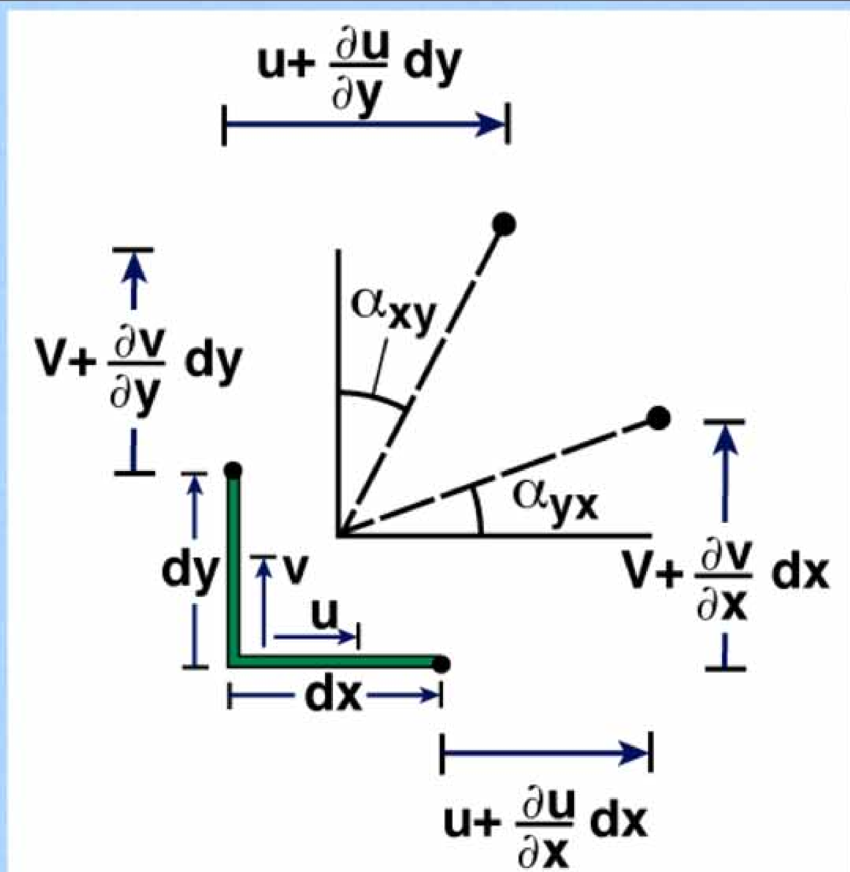
The displacements of points B_1 and C_1 can be approximated by

$$\boxed{\left(u + \frac{\partial u}{\partial x}dx\ ,\ v + \frac{\partial v}{\partial x}dx\right),}$$

and

$$\boxed{\left(u + \frac{\partial u}{\partial y}dy\ ,\ v + \frac{\partial v}{\partial y}dy\right)}$$





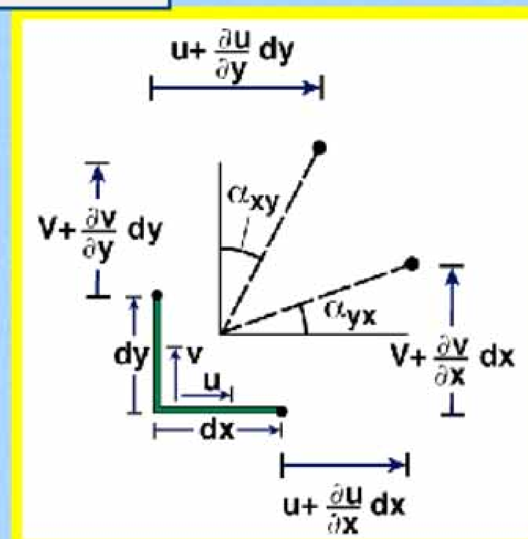
Strain-Displacement Relationships

The linear (extensional) strain in the x direction

$$\varepsilon_x = \frac{\text{change in the projected length (on the x–axis) of element dx}}{\text{original length}}$$

$$= \frac{u + \frac{\partial u}{\partial x}dx - u}{dx}$$

$$= \frac{\partial u}{\partial x}$$



Strain-Displacement Relationships

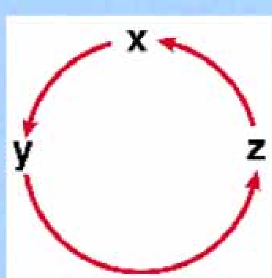
The linear (extensional) strain in the x direction

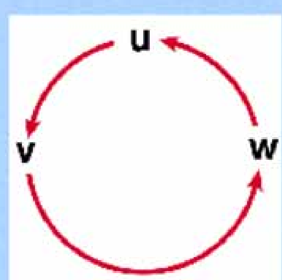
$$\varepsilon_x = \frac{\text{change in the projected length (on the x–axis) of element dx}}{\text{original length}}$$

Analogously, linear strains in y and z directions are given by:

$$\varepsilon_y = \frac{\partial v}{\partial y}$$

$$\varepsilon_z = \frac{\partial w}{\partial z}$$





Strain-Displacement Relationships

The shearing strain in the plane x-y is defined as the change in the angle between AB and AC

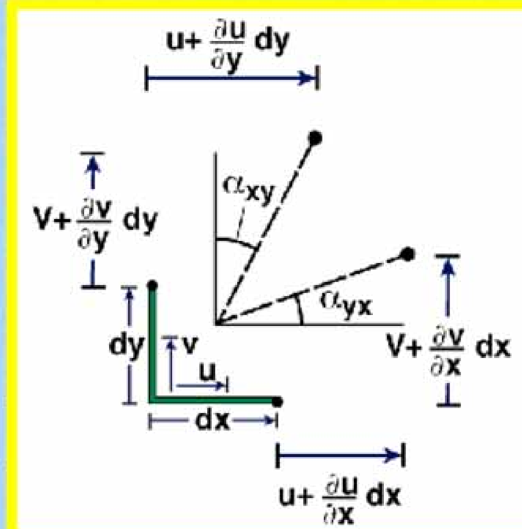
γ_{xy} = shearing strain in the plane xy

$$= \alpha_{xy} + \alpha_{yx}$$

For small displacement gradients and small strains

$$\alpha_{yx} \simeq \tan \alpha_{yx}$$

$$= \frac{v + \frac{\partial v}{\partial x} dx \ - v}{dx + \frac{\partial u}{\partial x} dx}$$



Strain-Displacement Relationships

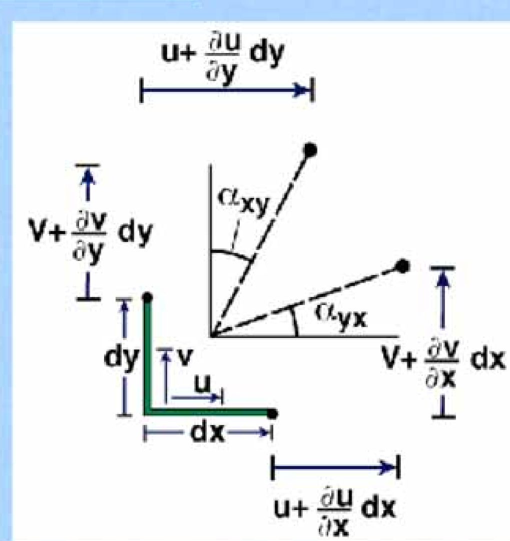
The shearing strain in the plane x-y is defined as the change in the angle between AB and AC

γ_{xy} = shearing strain in the plane xy

$$= \alpha_{xy} + \alpha_{yx}$$

For small displacement gradients and small strains

$$= \frac{\frac{\partial v}{\partial x}}{1 + \frac{\partial u}{\partial x}}$$



Strain-Displacement Relationships

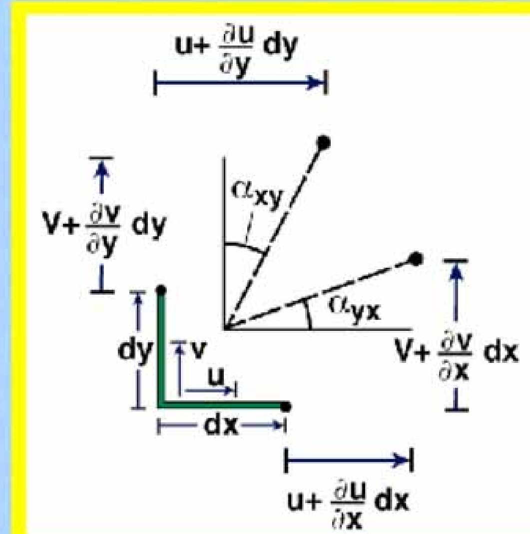
The shearing strain in the plane x-y is defined as the change in the angle between AB and AC

γ_{xy} = shearing strain in the plane xy

$= \alpha_{xy} + \alpha_{yx}$

For small displacement gradients and small strains

$\cong \dfrac{\partial v}{\partial x}$



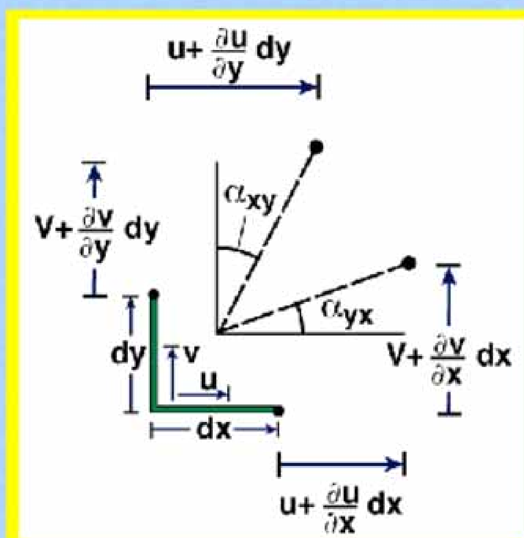
Strain-Displacement Relationships

Analogously

$\alpha_{xy} \simeq \dfrac{\partial u}{\partial y}$

From which

$\gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}$

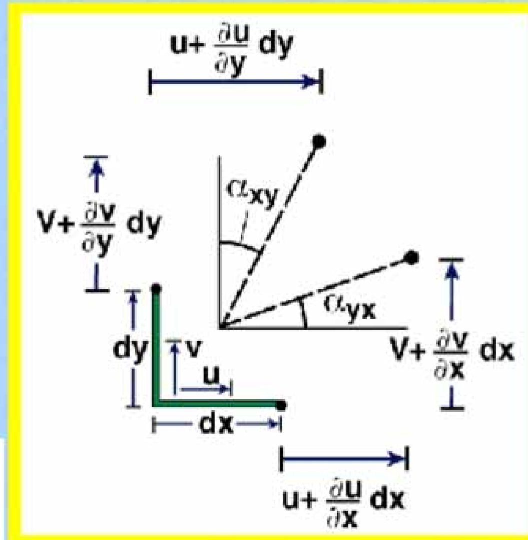


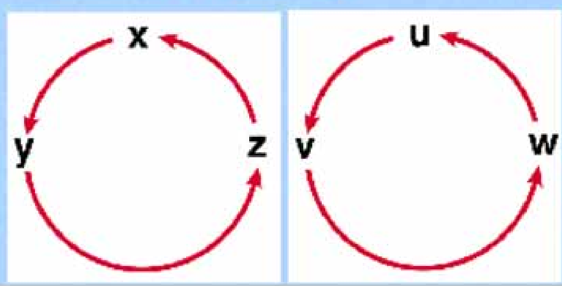
Strain-Displacement Relationships

- Analogously, the shearing strains in the planes yz and zx are given by:

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

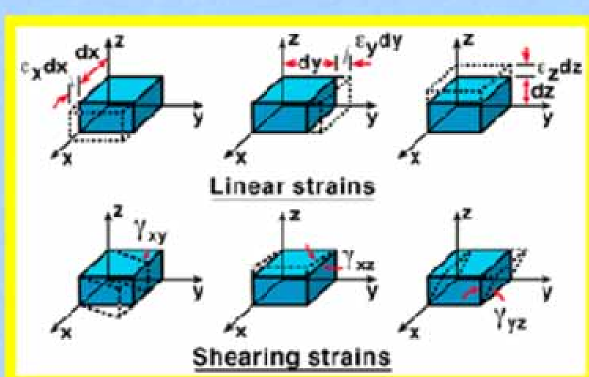
$$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$



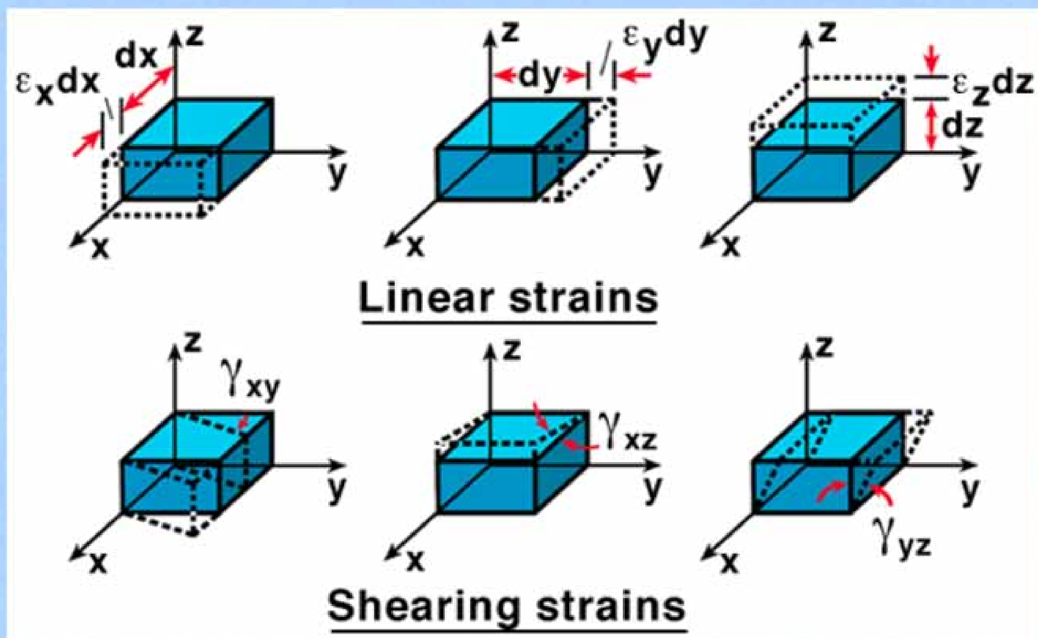


Strain-Displacement Relationships

Sign Convention



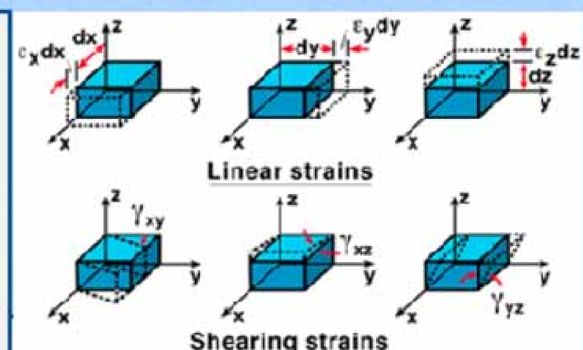
- Linear (extensional) strains are positive if tensile
- Shearing strains are positive when they decrease the angle between the sides



Strain-Displacement Relationships

Matrix Form of Strain-Displacement Relationships

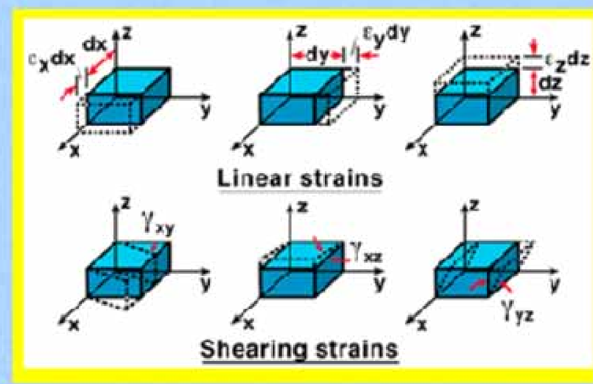
$$
\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \hline \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix}
=
\begin{bmatrix}
\frac{\partial}{\partial x} & \cdot & \cdot \\
\cdot & \frac{\partial}{\partial y} & \cdot \\
\cdot & \cdot & \frac{\partial}{\partial z} \\
\hline
\cdot & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\
\frac{\partial}{\partial z} & \cdot & \frac{\partial}{\partial x} \\
\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & \cdot
\end{bmatrix}
\begin{Bmatrix} u \\ v \\ w \end{Bmatrix}
$$



Strain-Displacement Relationships

Matrix Form of Strain-Displacement Relationships

Strain vector = $\{\varepsilon\}$

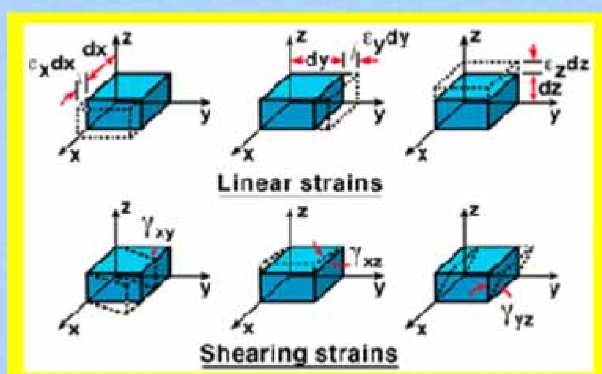


$$\{\varepsilon\}^t = \left[\varepsilon_x \ \varepsilon_y \ \varepsilon_z \ \middle| \ \gamma_{yz} \ \gamma_{zx} \ \gamma_{xy}\right]$$

Strain-Displacement Relationships

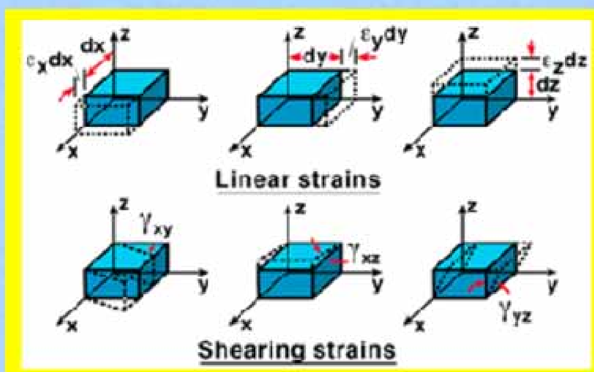
Notes

- Strains are defined at a point - an infinitesimal volume element
- Linear (extensional) strains are associated with:
  - Change in the volume of the element
  - Change in the shape (or form) of the element (elemental cube is transformed into a rectangular parallelopiped)



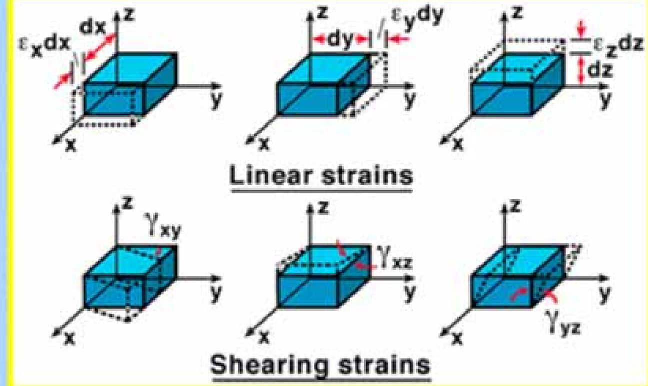
Strain-Displacement Relationships

- Linear (extensional) strains are associated with:
  - Change in the volume of the element
  - Change in the shape (or form) of the element (elemental cube is transformed into a rectangular parallelopiped)
  - Shearing strains are associated with change in the shape (or form) of the element



Analysis of Strain

- The transformation of strain components (associated with coordinate transformations), the determination of principal strains, principal directions, maximum shearing strains and octahedral strains follow similar procedures to those used for stresses.
- The equations for stresses can be used for strains if the following substitutions are made:

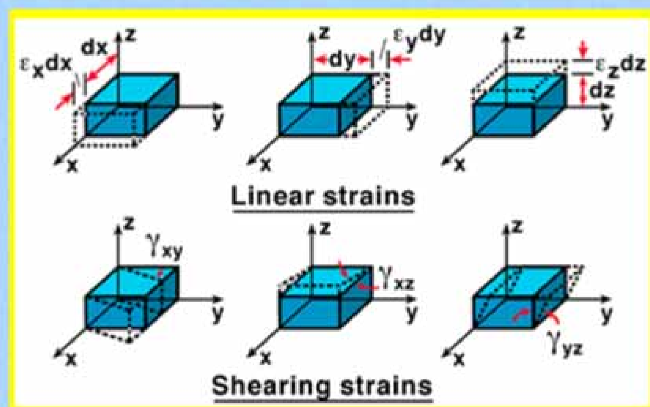


Analysis of Strain

- The equations for stresses can be used for strains if the following substitutions are made:

$$\varepsilon_x \leftrightarrow \sigma_{xx} \qquad \varepsilon_y \leftrightarrow \sigma_{yy} \qquad \varepsilon_z \leftrightarrow \sigma_{zz}$$

$$\frac{1}{2}\gamma_{yz} \leftrightarrow \tau_{yz}$$
$$\frac{1}{2}\gamma_{zx} \leftrightarrow \tau_{zx}$$
$$\frac{1}{2}\gamma_{xy} \leftrightarrow \tau_{xy}$$

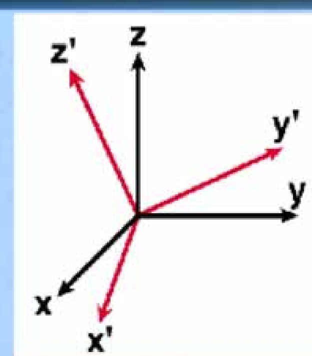


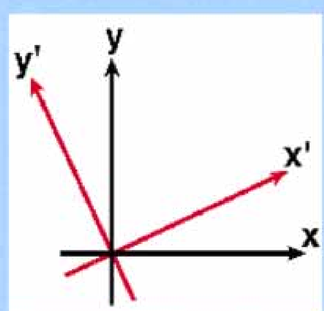
Transformation of Strain Components

$$\left[\varepsilon'\right] = \left[T\right]\left[\varepsilon\right]\left[T\right]^T$$

where

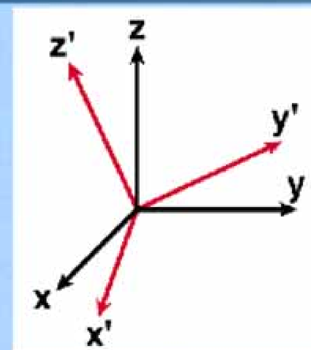
$$\left[\varepsilon'\right] = \begin{bmatrix} \varepsilon_{x'} & \frac{1}{2}\gamma_{x'y'} & \frac{1}{2}\gamma_{x'z'} \\ & \varepsilon_{y'} & \frac{1}{2}\gamma_{y'z'} \\ \text{Symm.} & & \varepsilon_{z'} \end{bmatrix}$$





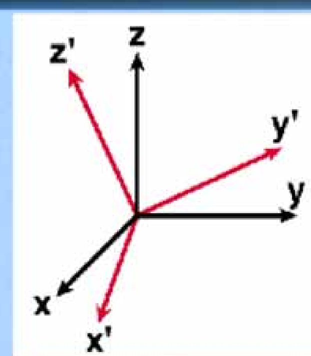
Transformation of Strain Components

$$[\varepsilon] = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ & & \varepsilon_z \end{bmatrix}$$



Transformation of Strain Components

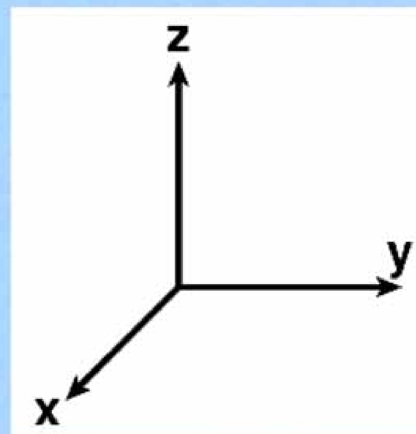
$$[T] = \begin{bmatrix} \ell & m_1 & n_1 \\ \ell & m_2 & n_2 \\ \ell & m_3 & n_3 \end{bmatrix}$$



Principal Strains and Principal Directions

Solution of an algebraic eigenvalue problem

$$\begin{vmatrix} \varepsilon_x-\varepsilon & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y-\varepsilon & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{xz} & \varepsilon_z-\varepsilon \end{vmatrix} \begin{Bmatrix} \ell \\ m \\ n \end{Bmatrix} = 0$$



with $\ell^2 + m^2 + n^2 = 1$

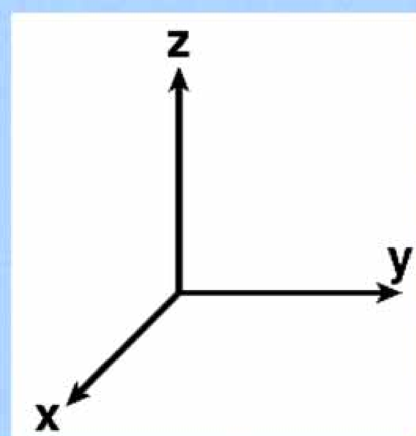
Characteristic equation

$$-\varepsilon^3 + J_1\varepsilon^2 - J_2\varepsilon + J_3 = 0$$

where $J_1 = \varepsilon_x + \varepsilon_y + \varepsilon_z$

Principal Strains and Principal Directions

$$J_2 = \begin{vmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y \end{vmatrix} + \begin{vmatrix} \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{vmatrix} + \begin{vmatrix} \varepsilon_z & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xz} & \varepsilon_x \end{vmatrix}$$



$$J_3 = \begin{vmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{vmatrix}$$

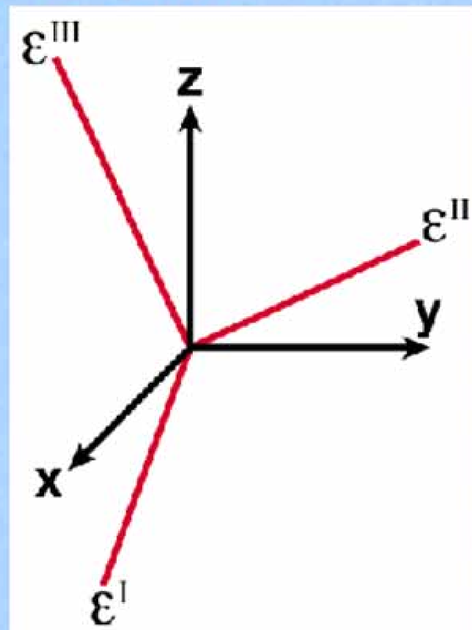
Principal Strains and Principal Directions

Principal strains and principal directions

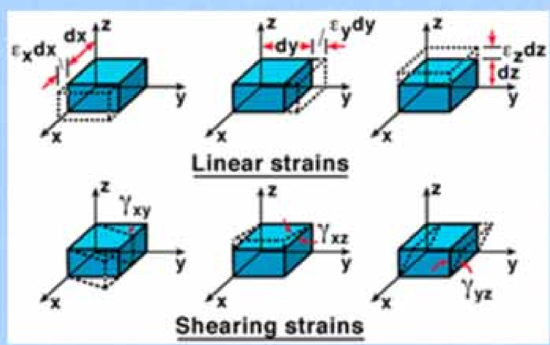
$$\varepsilon^{I} \to \left(\ell^{I}, m^{I}, n^{I}\right)$$

$$\varepsilon^{II} \to \left(\ell^{II}, m^{II}, n^{III}\right)$$

$$\varepsilon^{III} \to \left(\ell^{III}, m^{III}, n^{III}\right)$$

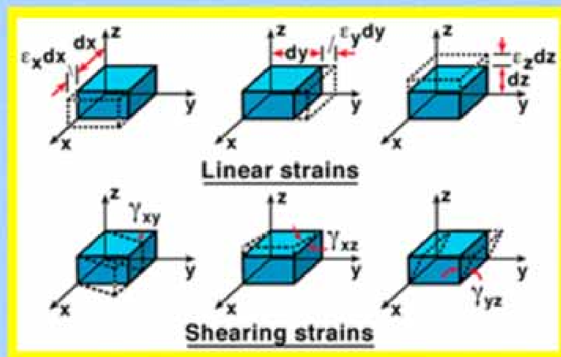


Principal Strains and Principal Directions



$$\underbrace{\begin{bmatrix} \varepsilon_x & \frac{\gamma_{xy}}{2} & \frac{\gamma_{xz}}{2} \\ \frac{\gamma_{xy}}{2} & \varepsilon_y & \frac{\gamma_{yz}}{2} \\ \frac{\gamma_{xz}}{2} & \frac{\gamma_{yz}}{2} & \varepsilon_z \end{bmatrix}} = \underbrace{\begin{bmatrix} \varepsilon_x - \frac{1}{3}J_1 & \frac{\gamma_{xy}}{2} & \frac{\gamma_{xz}}{2} \\ \frac{\gamma_{xy}}{2} & \varepsilon_y - \frac{1}{3}J_1 & \frac{\gamma_{yz}}{2} \\ \frac{\gamma_{xz}}{2} & \frac{\gamma_{yz}}{2} & \varepsilon_z - \frac{1}{3}J_1 \end{bmatrix}} + \underbrace{\begin{bmatrix} \frac{1}{3}J_1 & 0 & 0 \\ 0 & \frac{1}{3}J_1 & 0 \\ 0 & 0 & \frac{1}{3}J_1 \end{bmatrix}}$$

Principal Strains and Principal Directions



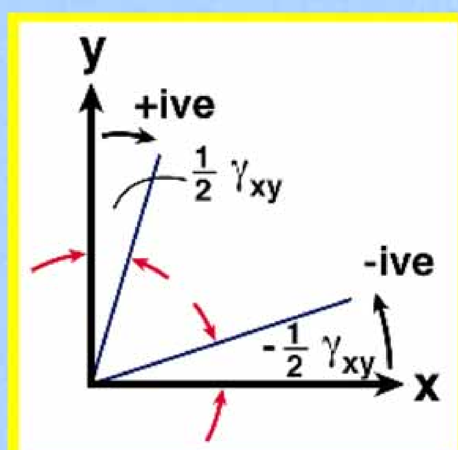
where J_1 = first strain invariant
$= \varepsilon x + \varepsilon y + \varepsilon z$

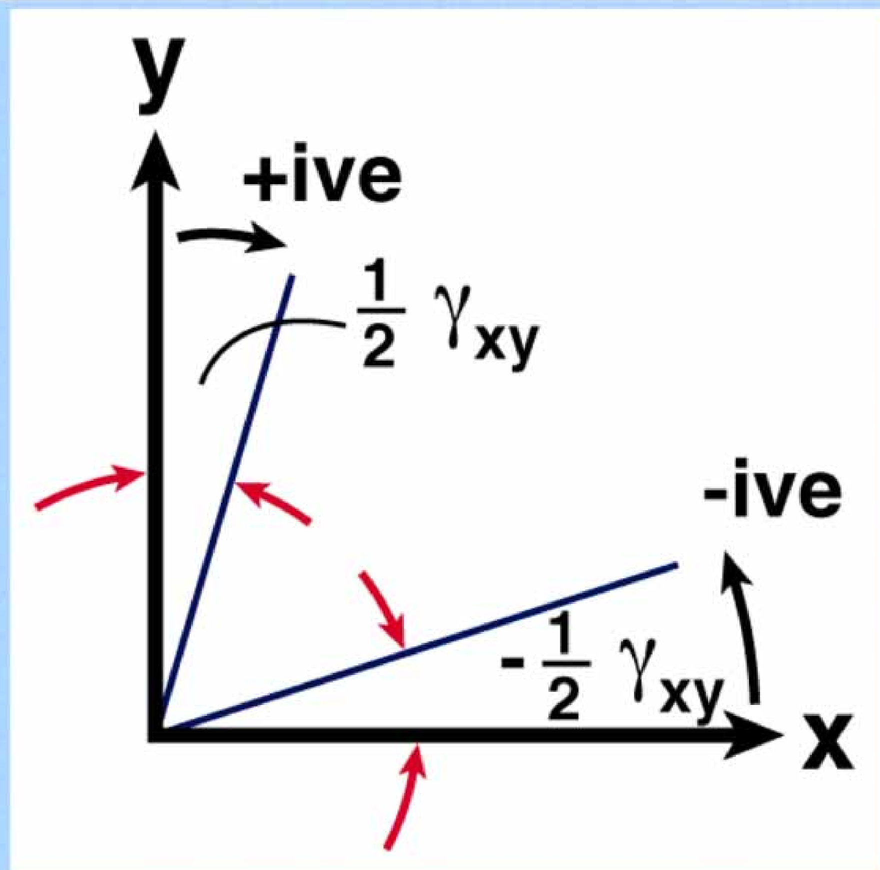
Plane Strain

A plane strain state, parallel to x-y, is said to exist if:

$$\boxed{\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0}$$

$$[\varepsilon] = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & 0 \\ \frac{1}{2}\gamma_{xy} & \varepsilon_x & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



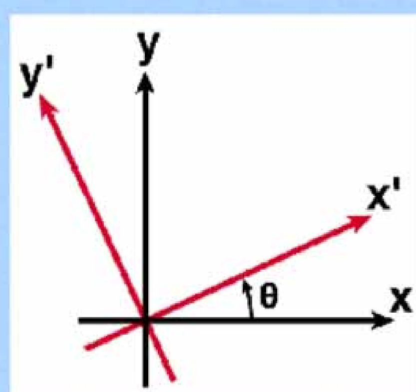


Plane Strain

Principal strains

$$\left\{\begin{matrix}\varepsilon^{I}\\ \varepsilon^{II}\end{matrix}\right\} = \frac{1}{2}\left(\varepsilon_x + \varepsilon_y\right) \pm \frac{1}{2}\sqrt{\left(\varepsilon_x - \varepsilon_y\right)^2 + \gamma_{xy}^2}$$

$$\tan 2\theta = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}$$

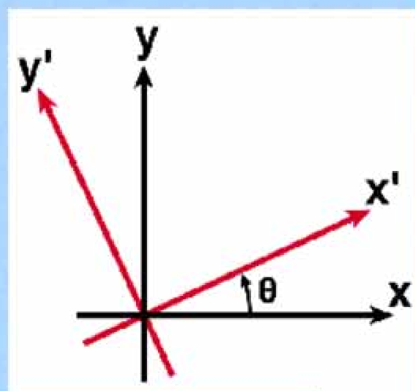


Maximum shearing strain

$$\gamma_{xy} = \pm\left(\varepsilon^{I} - \varepsilon^{II}\right)$$

Plane Strain

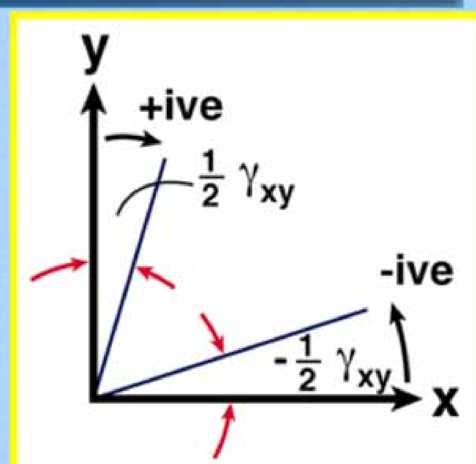
Transformation of Strain Components



$$\begin{Bmatrix} \varepsilon_{x'} \\ \varepsilon_{y'} \\ \frac{1}{2}\gamma_{x'y'} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & 2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta-\sin^2\theta \end{bmatrix} \begin{Bmatrix} \varepsilon_{x'} \\ \varepsilon_{y'} \\ \frac{1}{2}\gamma_{x'y'} \end{Bmatrix}$$

Mohr's Circle Representation of Plane Strain

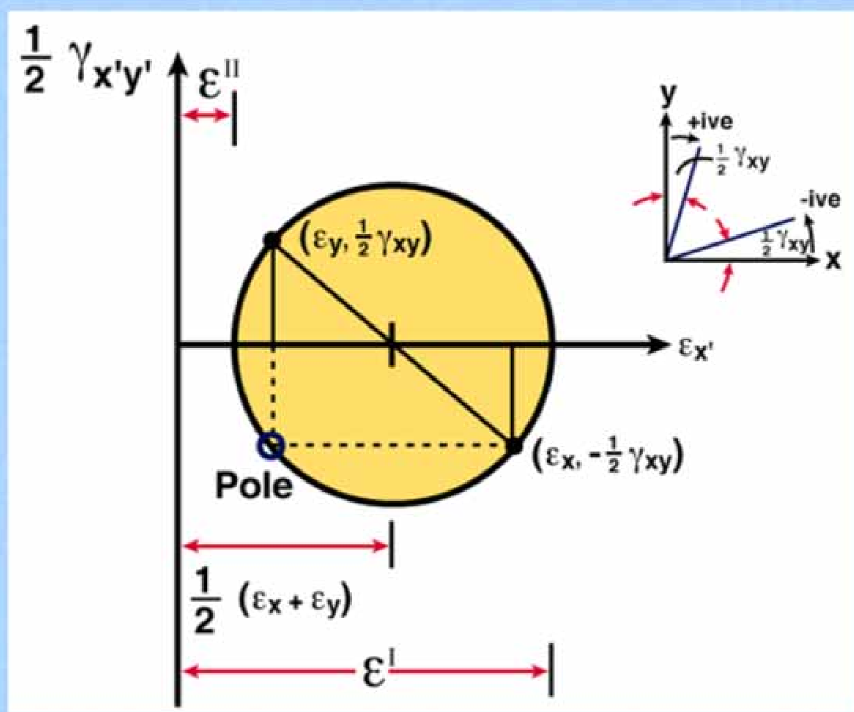
Sign Convention



- Linear (extensional) strain is positive when tensile
- Shearing strain
  - if γ_{xy} is positive then
    - $\frac{1}{2}\gamma_{xy}$ with ε_x is counterclockwise, taken as negative
    - $\frac{1}{2}\gamma_{xy}$ with ε_y is clockwise, taken as positive

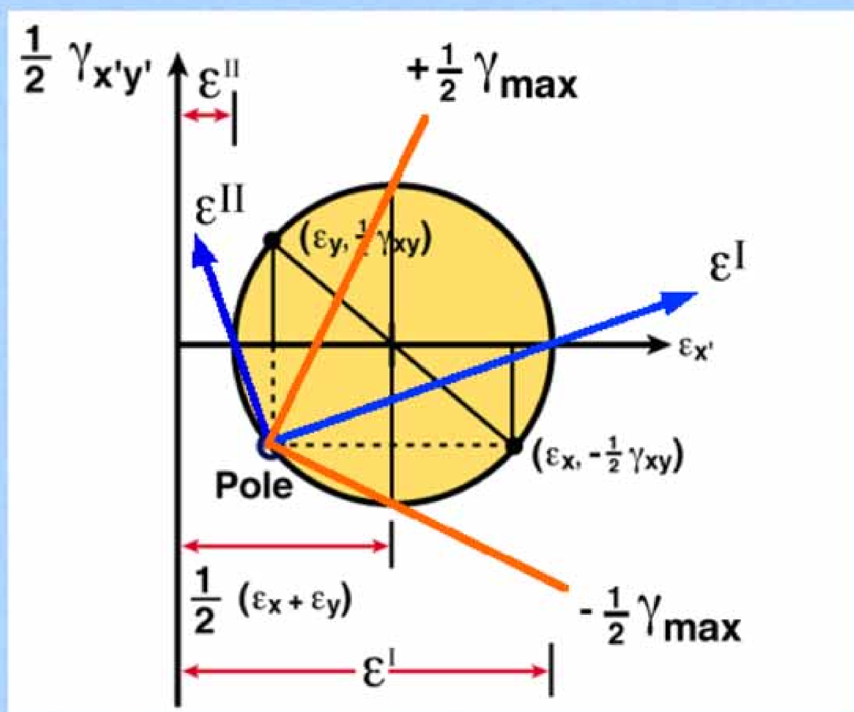
Mohr's Circle Representation of Plane Strain

Pole



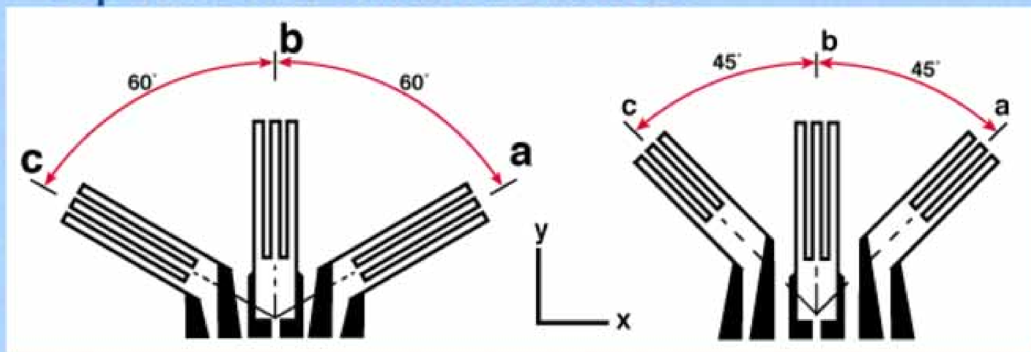
Mohr's Circle Representation of Plane Strain

Pole



Strain Measurements

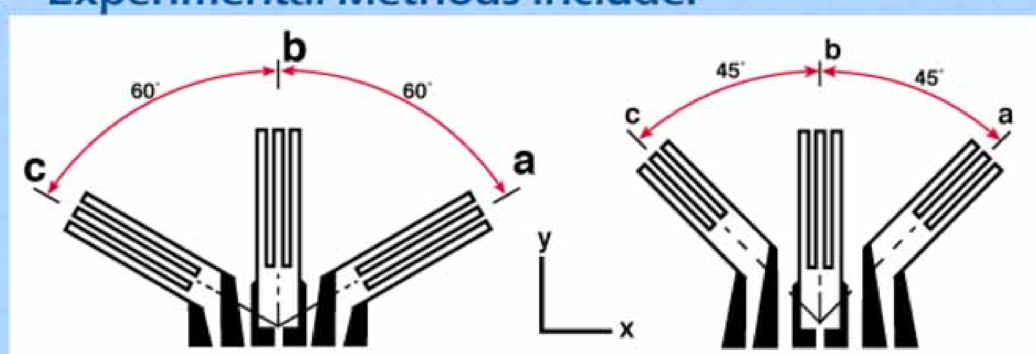
- Experimental Methods include:



- Electrical resistance (bonded) strain gages
  - measure extensional strains (extension / contraction) of lines on the surface of a member

Strain Measurements

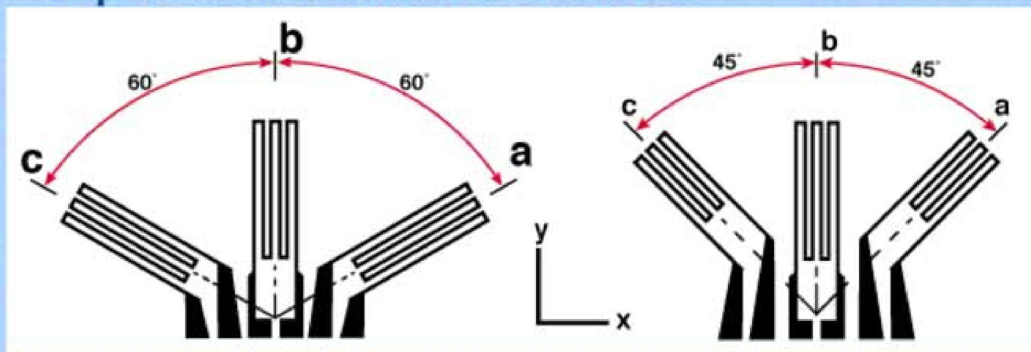
- Experimental Methods include:



- It is customary to cluster three gages (strain rosettes)
  - Delta rosette (with gages spaced at 60° angles)

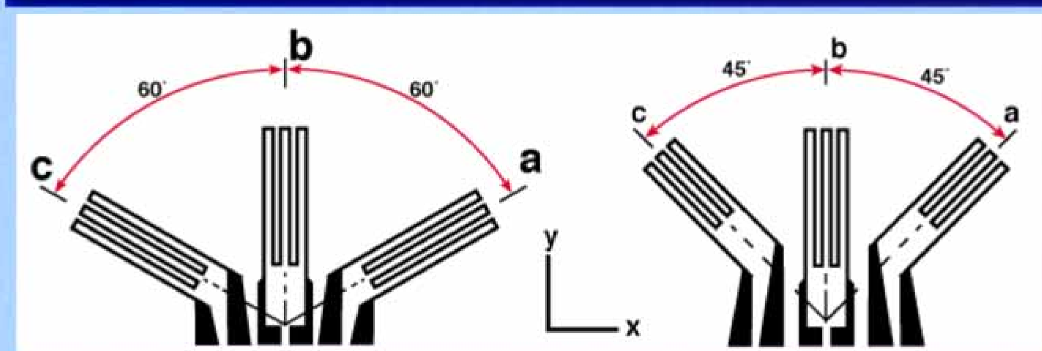
Strain Measurements

- Experimental Methods include:



  - It is customary to cluster three gages (strain rosettes)
    - Rectangular rosette (with gages spaced at 45° angles)

Strain Measurements



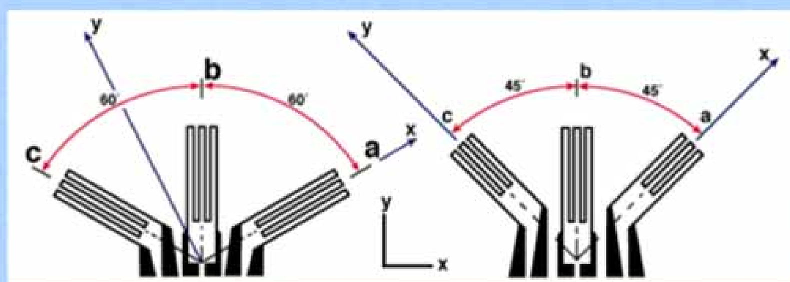
- Photoelastic methods
- Holographic
- Moire'
- Speckle interferometry techniques

Strain Measurements

Strain Rosettes

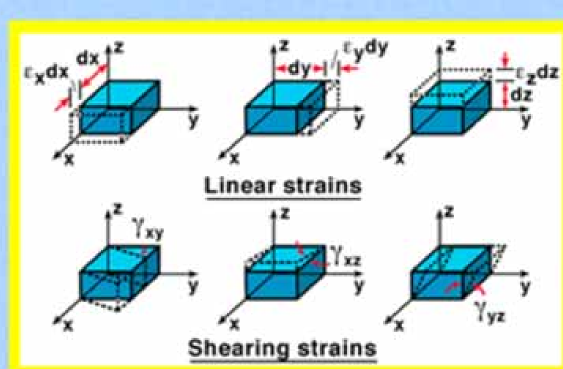
$$\begin{Bmatrix} \varepsilon_a \\ \varepsilon_b \\ \varepsilon_c \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \cos^2\theta & \sin^2\theta & \sin\theta\cos\theta \\ \cos^2 2\theta & \sin^2 2\theta & \sin 2\theta \cos 2\theta \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

If ε_a, ε_b and ε_c are known, then $\varepsilon_x, \varepsilon_y, \gamma_{xy}$ can be found.



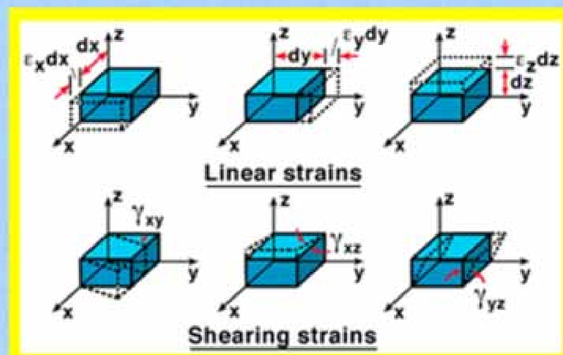
Strain Compatibility Relations

Strain-displacement relations have six strain components (ε_x, ε_y, ε_z γ_{yz}, γ_{zx}, γ_{xy}) and three displacement components (u, v, w).



Strain Compatibility Relations

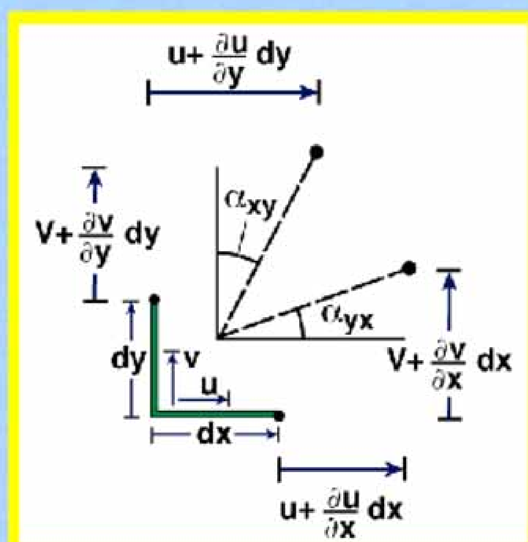
The three displacement components cannot be determined by integrating the six strain displacement relations. Certain relations among the strain components must exist in order to obtain the three displacement components.



Strain Compatibility Relations

For a plane strain case parallel to the x-y plane

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & \cdot \\ \cdot & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$



Shearing strains

$$\boxed{\frac{\partial^2 \varepsilon_x}{\partial y^2} = \frac{\partial^3 u}{\partial x \, \partial y^2}}$$

$$\boxed{\frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^3 v}{\partial x^2 \, \partial y}}$$

Strain Compatibility Relations

Shearing strains

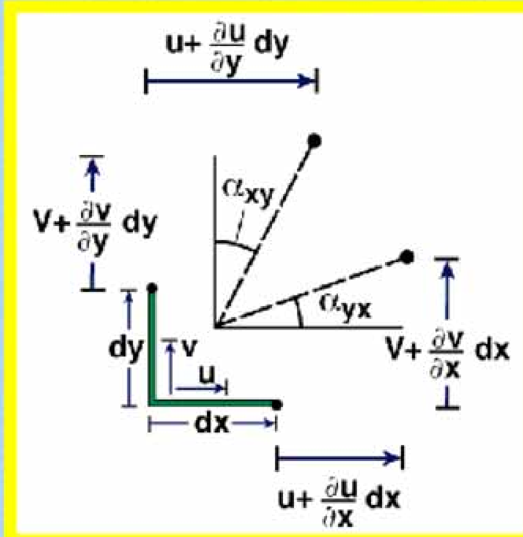
$$\frac{\partial^2 \varepsilon_x}{\partial y} = \frac{\partial^3 u}{\partial x\, \partial y^2}$$

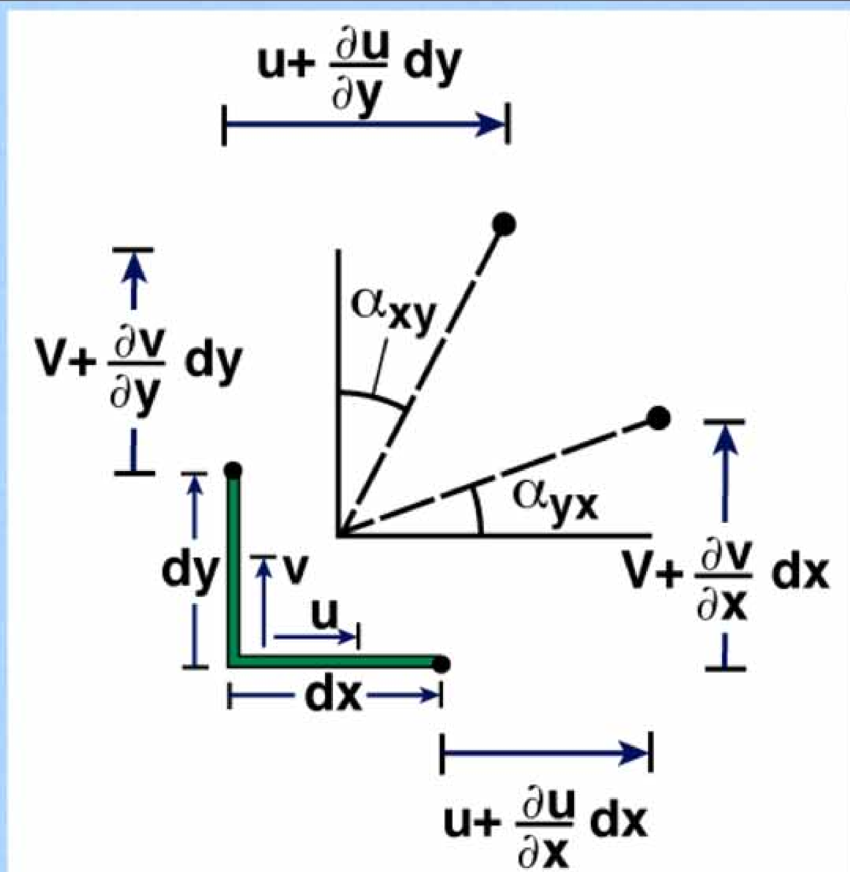
$$\frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^3 v}{\partial x^2\, \partial y}$$

$$\frac{\partial^2 \gamma_{xy}}{\partial x\, \partial y} = \frac{\partial^3 u}{\partial x\, \partial y^2} + \frac{\partial^3 v}{\partial x^2\, \partial y}$$

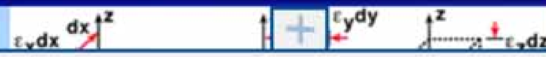
Therefore

$$\frac{\partial^2 \varepsilon_x}{\partial x\, \partial y} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x\, \partial y} = 0$$





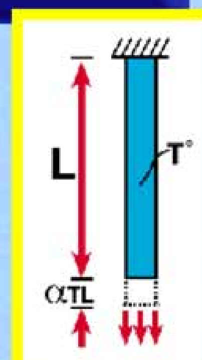
Strain Compatibility Relations



$$\left[\begin{array}{ccc|ccc}
\bullet & \frac{\partial^2}{\partial z^2} & \frac{\partial^2}{\partial y^2} & -\frac{\partial^2}{\partial y \partial z} & \bullet & \bullet \\
\frac{\partial^2}{\partial z^2} & \bullet & \frac{\partial^2}{\partial x^2} & \bullet & -\frac{\partial^2}{\partial z \partial x} & \bullet \\
\frac{\partial^2}{\partial y^2} & \frac{\partial^2}{\partial x^2} & \bullet & \bullet & \bullet & -\frac{\partial^2}{\partial x \partial y} \\
\hline
\frac{\partial^2}{\partial y \partial z} & \bullet & \bullet & \frac{1}{2}\frac{\partial^2}{\partial x^2} & -\frac{1}{2}\frac{\partial^2}{\partial x \partial y} & -\frac{1}{2}\frac{\partial^2}{\partial x \partial z} \\
\bullet & \frac{\partial^2}{\partial z \partial x} & \bullet & -\frac{1}{2}\frac{\partial^2}{\partial y \partial x} & \frac{1}{2}\frac{\partial^2}{\partial y^2} & -\frac{1}{2}\frac{\partial^2}{\partial y \partial z} \\
\bullet & \bullet & \frac{\partial^2}{\partial x \partial y} & -\frac{1}{2}\frac{\partial^2}{\partial z \partial x} & -\frac{1}{2}\frac{\partial^2}{\partial z \partial y} & \frac{1}{2}\frac{\partial^2}{\partial z^2}
\end{array}\right]
\left\{\begin{array}{c}
\varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \hline \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy}
\end{array}\right\} = 0$$

Thermal Strains

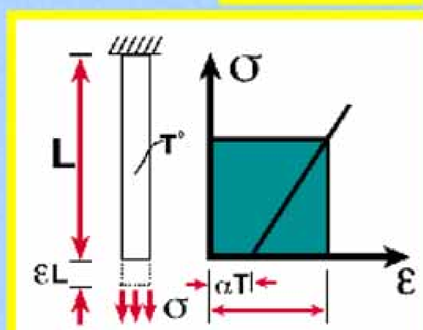
- Under uniform temperature change T°
  elongation of bar = α T L
  where α = coefficient of thermal expansion

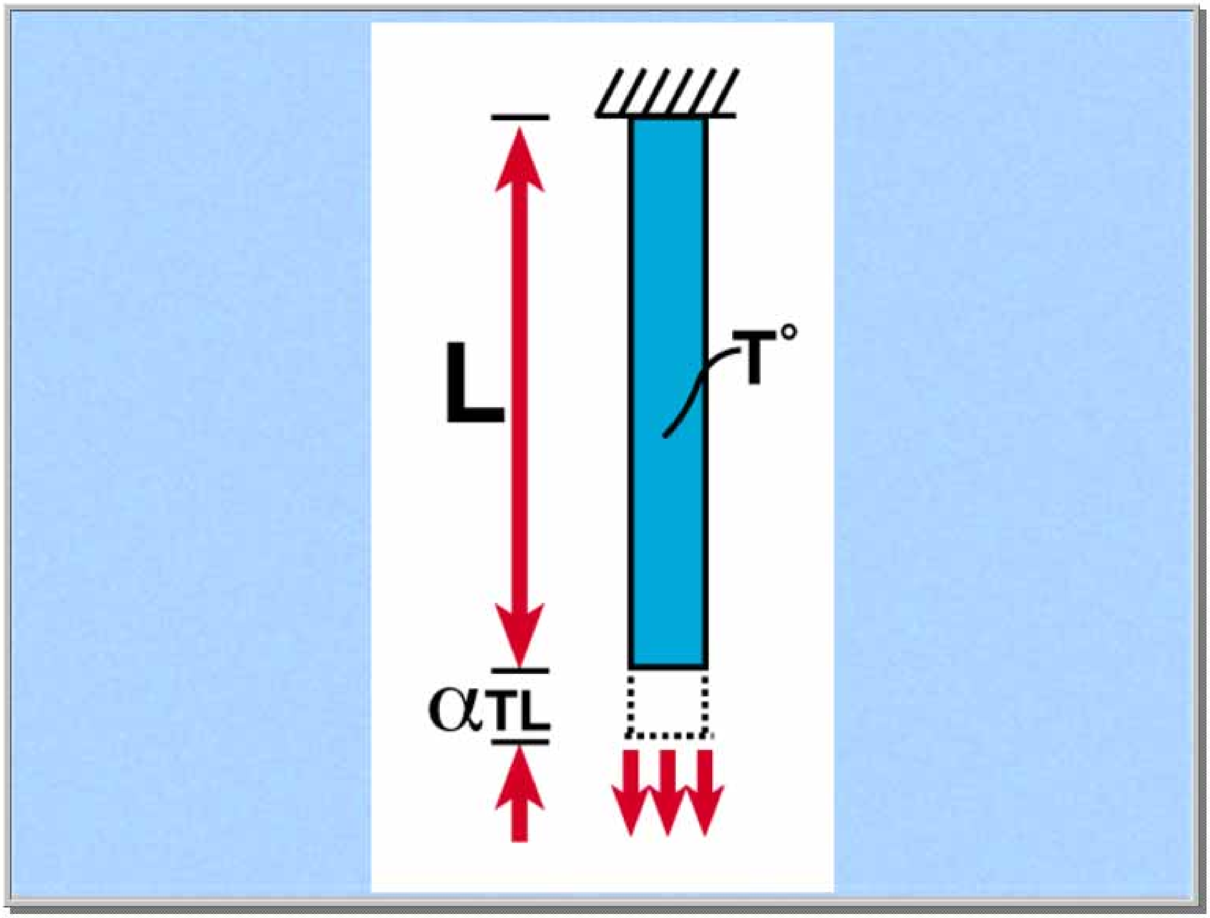


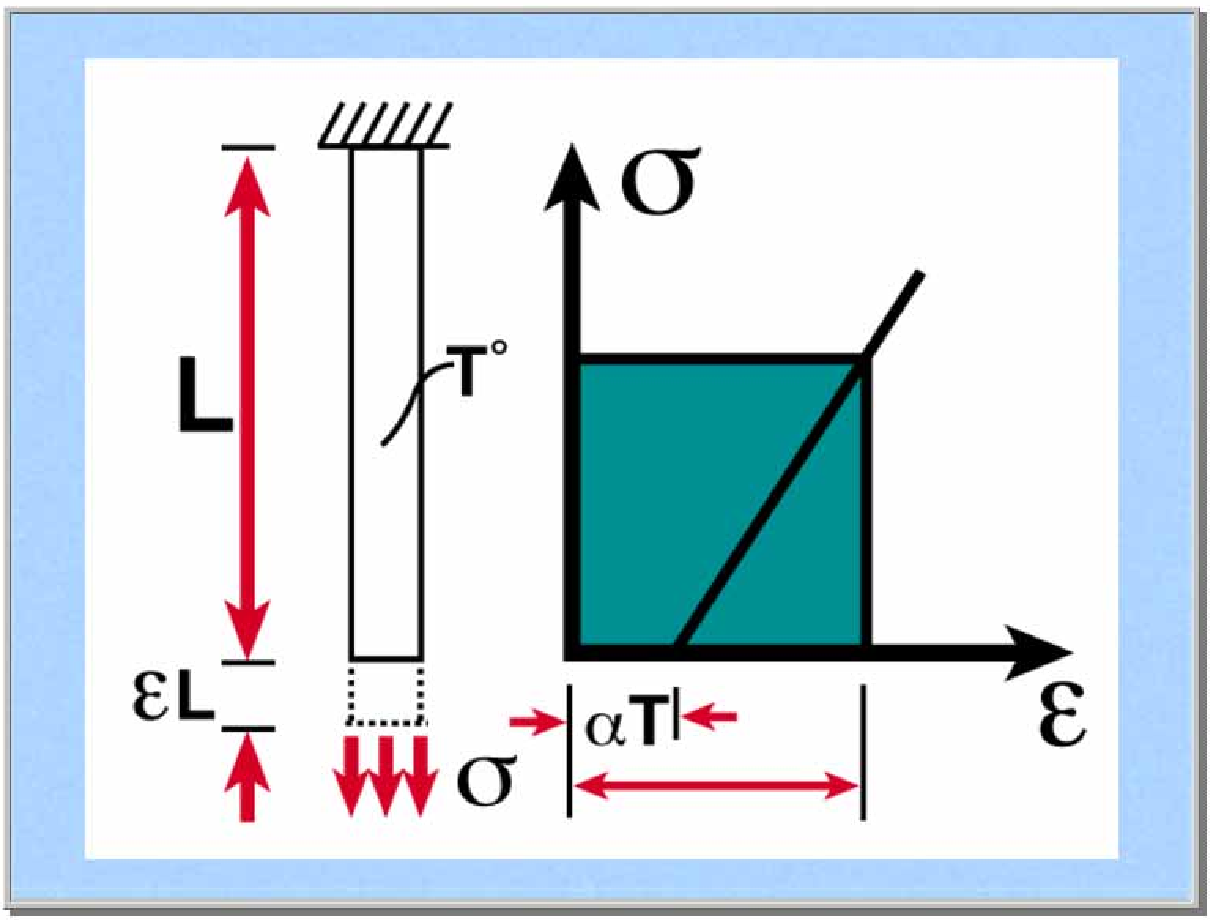
- Thermal strain = α T
  but thermal stress = 0
  since there is no resistance to the expansion

- For the case of combined mechanical and thermal strains

$$\boxed{\sigma = E\left(\varepsilon - \alpha T\right)}$$




L
T°
αTL


σ
L
T°
εL
σ
αT
ε